Bezout’s theorem: I
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This week we will be studying the tropical analogue of Bezout’s theorem, which in classical geometry gives a precise count of the number of intersections between two projective algebraic curves. It turns out, interestingly, that although all the definitions (except that of intersection) are different in tropical geometry, we can also prove a theorem with the same statement. Next time, we will see that in fact the two theorems are closely related.

1 Classical Bezout

Let me begin by reminding you of the statement of Bezout’s theorem for algebraic curves:

**Theorem 1.** Let $C$ and $D$ be algebraic curves in $\mathbb{CP}^2$, having degrees $d$ and $e$. At every point $p$ of $C \cap D$ we define a positive integer $m_p$, the intersection multiplicity there, which depends only on the properties of $C$ and $D$ which are local to $p$. If the curves are smooth at $p$ and their tangent lines there are different (in which case $C$ and $D$ intersect transversely there), then $m_p = 1$. Having done this, we conclude that

$$\sum_{p \in C \cap D} m_p = d \cdot e.$$ 

In particular, if $C$ and $D$ intersect transversely, then $\#(C \cap D) = de$.

The definition of intersection multiplicity for algebraic curves can be quite involved, although if one of $C$ and $D$ is smooth, then one can give a natural definition: using, say, the tangent line to $C$, we can give coordinates to $C$ in a region around $p$; suppose $g(X, Y, Z) = 0$ is the equation of $D$ in $\mathbb{CP}^2$, so that $g$ gives a function on $C$ which vanishes at $p$. Since we have given coordinates on $C$ around $p$, this means that locally we are looking at a polynomial function on $C$ with a zero at 0. The intersection multiplicity is the multiplicity of this zero as a root of $g|_C$. When neither $C$ nor $D$ is smooth, one has to abandon such geometric intuition and use commutative algebra (for those “in the know”, if $f$ and $g$ are the polynomials defining $C$ and $D$, the intersection multiplicity at $p = (X_0, Y_0, Z_0)$ is defined as: let $\mathcal{O}_p = \mathbb{C}[X, Y, Z]/\mathfrak{m}$, where $\mathfrak{m} = (X - X_0, Y - Y_0, Z - Z_0)$ is the maximal ideal corresponding to $p$, and let $A = \mathbb{C}[X, Y, Z]/(f, g)$ be the algebra corresponding to the intersection $C \cap D$; it contains the quotient $\mathfrak{m} = \mathfrak{m}/(f, g)$, since the equation $f(X_0, Y_0, Z_0) = g(X_0, Y_0, Z_0) = 0$ means that $f, g \in \mathfrak{m}$. Then $A_{\mathfrak{m}}$ is a module over $\mathcal{O}_p$, and it is in fact both Artinian (no infinitely shrinking chains of submodules) and Noetherian (no infinitely growing chains), so its length (the number of links in a maximal chain of submodules) is finite. The intersection multiplicity is this length. You see why I didn’t want to give this definition?).

The intersection multiplicity is a necessary ingredient, since without it one can easily cook up curves with fewer than the required number of intersection points: for example, a line tangent to a second-degree curve will intersect it only once (but by the above definition of intersection multiplicity, that one point will count twice). But more importantly, it allows us to expand the scope of a particularly useful rule: if two curves $C$ and $D$ have $n$ points of intersection and you “tweak” $C$ by altering the coefficients of its equation by a small amount, then the resulting curve $C'$ also intersects $D$ in $n$ points. This rule is always true when $C$ and $D$ intersect transversely (if you have taken differential topology, this statement might be familiar to you), but
is quite false when they share tangent lines. Take, for example, the line tangent to a quadratic curve, which goes from one intersection to two when it is tweaked, no matter how slightly.

It is also important that these curves be in \( \mathbb{CP}^2 \), rather than in \( \mathbb{RP}^2 \), since in that case it is possible to have no intersections at all, from which no definition of intersection multiplicity can rescue your count (again, the line tangent to the quadric curve can be tweaked away from it, for example the \( x \)-axis from a parabola \( y = x^2 \)). Strangely, this is not at all important in tropical geometry, though since we will be solving only linear equations there, this should not come as too much of a surprise.

I have, in the first lecture, already given a description of how one might prove this theorem. The technique is quite direct, and is simply to change the intersection into something that is easier to count (in this case, by replacing one curve with a collection of lines with the same overall degree). This seems to be a bit of a roundabout way of dealing with intersection multiplicity, though, since that is such a local quantity and would apparently change drastically when the curves are replaced; this is dealt with using a bit of algebraic machinery.

In tropical geometry, we won’t need to play this trick: we will count intersections by combining a number of local computations into one global result.

2 Tropical Bezout: the transverse case

In investigating Bezout’s theorem in tropical geometry, we should obviously expect the word “projective” to be important, so we will eventually be requiring that our curves be tropical projective curves; however, some setup at the beginning doesn’t need this. So let \( C \) and \( D \) be two curves, defined by tropical polynomials \( f(x, y) \) and \( g(x, y) \). To depict their intersections, we will necessarily also depict the rest of \( C \) and \( D \), and it turns out to be useful to consider \( C \cup D \) as itself a single tropical curve. The proof that it is tropical is quite simple:

**Proposition 2.** If \( C \) and \( D \) are two tropical curves, then so is \( C \cup D \). If they are defined by \( f(x, y) \) and \( g(x, y) \) then \( C \cup D \) is defined by \( f \circ g \). If \( C \) and \( D \) are projective of degrees \( d \) and \( e \), then \( C \cup D \) is projective of degree \( d + e \).

**Proof.** The proof of the second statement (which implies the first) is quite easy: simply observe that \( C \) and \( D \) are the loci of corners of the graphs of \( f \) and \( g \), or in other words, the set of points where these functions are not differentiable; since \( f \circ g \) is the sum of these two functions, a familiar theorem of calculus shows that it is smooth if and only if \( f \) and \( g \) both are; therefore its corner locus is the union of theirs. However, an independent proof of the first statement is instructive of what to expect with intersections.

As we know, a tropical curve is the same as a balanced graph, which means that \( C \) and \( D \) have edges of rational slope, satisfying a balancing condition at each vertex \( V \): if \( E_1, \ldots, E_n \) are the edges at \( V \), having multiplicities \( m_1, \ldots, m_n \), and \( \overrightarrow{u}_1, \ldots, \overrightarrow{u}_n \) are the primitive integral vectors pointing away from \( V \) along these edges, then

\[
\sum_{i=1}^{n} m_i \overrightarrow{u}_i = 0.
\]

Let us verify these facts for \( C \cup D \) (of course, we never did prove properly that a balanced graph is a tropical curve...but that is left for the projects). Since every edge of \( C \cup D \) is an edge of one of \( C \) or \( D \), they all have rational slopes. Let \( V \) be one of the vertices; then there are four situations:

1. \( V \) is a vertex of either \( C \) or \( D \), with no edges of the other passing through it. Then the balancing condition holds because it does in the original curves.

2. \( V \) is the intersection of an edge of \( C \) with an edge of \( D \), contained in the interior of both. Then it has four edges, which are opposed in pairs (pairs of which each member has equal multiplicity); therefore the balancing condition holds because the vectors cancel two at a time.
3. \( V \) is a vertex \( W \) of one curve (say \( C \)) with an edge \( E \) of the other (\( D \)) passing through it. This situation has elements of the first two cases in it: \( E \) contributes a pair of opposing vectors, and every other edge at \( V \) is an edge of \( C \) at \( W \), so their vectors are balanced by assumption.

4. \( V \) is a vertex of both \( C \) and \( D \); then the edges break down into those which are edges of \( C \) and those which are edges of \( C \), which are independently balanced.

To be especially precise, we have to note that it is possible for an edge of \( C \) to partially coincide with an edge of \( D \) (the region where they overlap must end in vertices of \( C \cup D \), since it marks a place where the edges of either \( C \) or \( D \) change slope); in order for the division of labor we’ve done to work out, we must give this “stacked” edge the sum of the multiplicities of its constituents; thus, if two degree-1 edges stack, the result has degree 2.

To show the third statement, one could again argue using the polynomials \( f \) and \( g \), or using the graphs themselves. To use the polynomials, one observes that projectivity of degree \( n \) means that \( x^{\circ n}, y^{\circ n}, \) and \( 0 \) are all terms of the polynomial; thus, \( f \) has the appropriate terms of degree \( d \) and \( g \) of degree \( e \). Taking their tropical product, one sees that, just as in ordinary algebra, the terms \( x^{\circ (d+e)}, y^{\circ (d+e)}, \) and \( 0 \) are, respectively, the product of the terms \( x^{\circ d} \) and \( x^{\circ e} \), \( y^{\circ d} \) and \( y^{\circ e} \), and \( 0 \), of \( f \) and \( g \), which shows that they all appear in \( f \circ g \).

To use the graphs, the corresponding statement is that projectivity of degree \( n \) means that there are \( n \) edges (counted with multiplicity) going in the \((-1, 0), (0, -1) \) and \((1, 1) \) directions only. Thus, \( C \) has \( d \) of these, \( D \) has \( e \) of these, and \( C \cup D \) has the union of them all, which, according to our multiplicity convention above, means that even if some of them coincide, there are \( d + e \) of them.

Knowing this, the proof of Bezout’s theorem writes itself. Consider that we are interested in intersection points of \( C \) and \( D \); they are places where the edges of these two curves cross, and therefore represent vertices of \( C \cup D \), which (as we have observed) there are four types. For now, we will confine ourselves to dealing with curves intersecting in only the second type of vertex: when \( C \) and \( D \) intersect only in the interiors of their edges; we will say that they are \emph{apparently transverse} there (the qualifier will be explained presently).

In that case, \( C \cup D \) has only the first two types of vertices, which break down as follows:

1. The vertices of \( C \);
2. The vertices of \( D \);
3. The points of \( C \cap D \).

These sets are mutually exclusive, which suggests that we can get some mileage out of considering the subdivided Newton polygon \( \mathcal{N}(C \cup D) \) of \( C \cup D \) and comparing it to those, \( \mathcal{N}(C) \) and \( \mathcal{N}(D) \), of \( C \) and \( D \). Indeed, the divisions of \( \mathcal{N}(C \cup D) \) correspond precisely to the vertices of \( C \cup D \), which are of the three above types. At the vertices of the first two types, the corresponding divisions are \emph{exactly the same polygons}, because the segments of the Newton polygon around a division are perpendicular to the edges of the tropical curve around the corresponding vertex, and their lengths can be recovered from the primitive integral vectors along those edges, and the multiplicities of the edges. Thus, \( \mathcal{N}(C \cup D) \) contains broken-up copies of \( \mathcal{N}(C) \) and \( \mathcal{N}(D) \), scattered among vertices of the third type, which are the intersection points lying among the original vertices. For each point \( p \) of \( C \cap D \), let \( P_p \) be its corresponding division of \( \mathcal{N}(C \cup D) \), which is a parallelogram since the intersection is apparently transverse. We have just argued that

\[
A(\mathcal{N}(C \cup D)) = A(\mathcal{N}(C)) + A(\mathcal{N}(D)) + \sum_{p \in C \cap D} A(P_p) = A(\mathcal{N}(C)) + A(\mathcal{N}(D)) + \sum_{p \in C \cup D} m_\mu(C, D) \tag{1}
\]

where \( A(\cdot) \) means the area of a plane figure and we have introduced the notation \( m_\mu(C, D) \) for the terms of the sum; these are the \emph{intersection multiplicities}.

Here, however, is where projectivity enters. If \( C \) and \( D \) are projective of degrees \( d \) and \( e \), then by Proposition 2, \( C \cup D \) is projective of degree \( d + e \), which means that their Newton polygons are each specific
right triangles in $\mathbb{Z}^2$. In particular, $N(C)$ has area $d^2/2$, $N(D)$ has area $e^2/2$, and $N(C \cup D)$ has area $(d+e)^2/2$. Doing the algebra, we find that

$$\sum_{p \in C \cap D} A(P_p) = de. \quad (2)$$

This is the “transverse” Bezout’s theorem for tropical projective curves.

Transversality is a misnomer, since the left side of this equation is not the number of intersection points, but a sum of various quantities which are not, apparently, even integers. These are the intersection multiplicities; even in an intersection which is apparently transverse (going by the configuration of the edges), they appear, unlike in the classical theorem. The concern about their nature is misplaced, as well: they are integers, as is easily seen. Indeed, let $v$ be one of the vertices of $P_p$, and let $\vec{v}_1$ and $\vec{v}_2$ be the two edges of $P_p$ at $v$; then $A(P_p) = ||\vec{v}_1 \times \vec{v}_2||$, as is well-known in linear algebra circles. These vectors, in turn, can be replaced by some defined intrinsically to $C$ and $D$: by the theorem on Newton subdivisions, the edges $E_i$ of $C \cup D$ corresponding to the $\vec{v}_1$ are orthogonal, and in fact $\vec{v}_i = m_i \vec{u}_i$, where the $\vec{u}_i$ are the primitive integral vectors along the $E_i$ pointing away from $p$. Therefore,

$$m_p(C, D) = m_1 m_2 ||\vec{u}_1 \times \vec{u}_2|| \quad (3)$$

is the intersection multiplicity at $p$. Since the $\vec{u}_i$ have integer coordinates, this number is a positive integer depending only on the local properties of $C$ and $D$ around $p$.

3 The stable intersection

This version of Bezout’s theorem is cute, but it requires too much of the curves. Although “most” curves will never intersect each others’ vertices, for those that do, we can provide an extension of the above computation. The main problem is to define the intersection multiplicities at “vertex-edge” or “vertex-vertex” intersections; the secondary problem is to take care of the even rarer situation which arose in the proof of Proposition 2, where entire segments of $C$ and $D$ coincided; in this situation, no definition of intersection multiplicity will prevent the sum of Equation 1 from being very, very infinite.

A simple example shows that it is not correct to make the same definition of multiplicity. One can find a tropical line $L$ and a tropical quadric $Q$ intersecting at a vertex of $L$ and an edge of $Q$ (having slope 1/2) and no other points:

The corresponding division of the Newton polygon is an irregular pentagon having area 2.5, whereas the intersection itself is supposed to have 2 total points. In fact, every example you can come up with involving a non-transverse intersection will have this “excess intersection” property. The reason is, philosophically, that at the trivalent vertex of $L$, a single edge of $Q$ will be intersecting all three incoming edges, whereas normally, it would only be able to cross one or two (depending on which side it passed). This is not precise for several reasons, but it suggests that the solution is to “tweak” the intersection so that the vertex is no longer a problem.

So, let $Q'$ be $Q$, shifted sightly in any direction, so that its edge no longer contains the vertex of $L$. Now we have one of the following Newton subdivisions, corresponding to one of the pictures:

Notice how, although the middle pentagon has shattered into several pieces (two in one case, three in the other), the other divisions are unaffected: their vertices are just too far from the vertex of $L$ to be affected by a small shift like this. We see that the actual intersections of $Q'$ with $L$ are the parallelograms with area each 1, or the large parallelogram with area 2 (depending on the picture). This is the correct number, and it doesn’t depend on the tweak we made, so long as it was both small enough to avoid altering the other vertices, but nonzero.

Now we can try to remove the tweak: as it approaches zero, we see that the transverse intersections slide in towards the vertex of $L$ along its edges, and merge when we arrive back at $Q$, illustrating that the non-transverse intersection $Q \cap L$ should be considered a double point, either via two single points merging, or via one double point sliding in along the bottom edge.
Now that we have come up with the tweaking trick, we can apply it to our other nemesis, the improper intersection containing an entire line segment. This could be, for example, the union of two badly-placed lines:

Notice how, again, the trapezoid of intersection in the Newton polygon has area 1.5, rather than 1. If we tweak one of the lines, the trapezoid breaks:

And again, as the perturbation shrinks away, the transverse intersections slide back towards the vertex. This leads us to make the following general statement:

**Theorem 3.** Let $C$ and $D$ be any two tropical curves. If $C'$ and $D'$ are two small perturbations which intersect transversely, then the limit (as the perturbation vanishes) of their intersection points is exactly the set of vertices of $C \cup D$, the sum of the multiplicities of the points of $C' \cap D'$ which converge to $V$ is independent of the perturbations, provided they are small. This intersection (including the multiplicity of the intersection points) is called the stable intersection of $C$ and $D$.

**Proof.** By the “limit of (the set of) intersection points” I mean: the points of $C' \cap D'$ can be divided up into those which are close to particular vertices $V$ of $C \cup D$, and in each of these groupings, the points each vary continuously as the perturbation shinks, converging individually towards $V$. The proof of this fact is obvious, to the point that giving a precise statement is difficult: $C'$ and $D'$ are small translates of $C$ and $D$ and are graphs with straight edges; their intersections are thus the intersections of various lines, and it is not hard to argue by simply solving for the equality of various linear polynomials that the solutions vary continuously and converge as claimed. The only tricky point is to verify that when we have a pair of overlapping edges, no points of $C' \cap D'$ converge fo a point inside the edge, for which it suffices to observe that in fact, since $C'$ and $D'$ intersect transversely, any point of $C' \cap D'$ must be on an edge other than the overlapping ones.

The more interesting question is to show that the multiplicities don’t depend on the direction of perturbation. For this, we make use of the observation derived empirically above: let $V$ be one of the vertices of $C \cup D$, with corresponding local Newton polygon $\mathcal{N}_V$: it is not divided at all, since there is only one vertex at $V$ (obviously). The edges through $V$ are either those in $C$ or those in $D$ (or both if there is an overlapping edge, but we can split that up by multiplicity), and these subsets respectively have local Newton polygons $\mathcal{N}_V(C)$ and $\mathcal{N}_V(D)$; it may be that either $C$ or $D$ does not have a vertex at $V$, in which case locally, it is just a single line, but that is also a tropical curve having a single line segment for a Newton polygon. $C'$ and $D'$ have the same local Newton polgons, and as observed above, the portion of $C' \cup D'$ near $V$ has the same Newton polygon $\mathcal{N}_V$, but subdivided differently (or rather, at all). Since $C'$ and $D'$ intersect transversely, we can use the basic Bezout equality (1) to count $C' \cap D'$ near $V$! We conclude:

$$
\sum_{p \text{ near } V} A(P_p) = A(\mathcal{N}_V) - A(\mathcal{N}_V(C)) - A(\mathcal{N}_V(D)).
$$

The right side does not depend on the perturbation, so we conclude that neither does the left. 

This gives the correct definition of intersection multiplicity for a general non-transverse intersection point: from the area of the corresponding division of $\mathcal{N}(C \cup D)$, one must subtract the areas of the divisions of $\mathcal{N}(C)$ and $\mathcal{N}(D)$ around the point of intersection. The general Bezout’s theorem follows:

**Corollary 4.** Let $C$ and $D$ be tropical projective curves of degrees $d$ and $e$, and let $S$ be their stable intersection, whose points $p$ have multiplicity $m_p(C, D)$ as defined above. Then

$$
\sum_{p \in S} m_p(C, D) = d \cdot e.
$$

**Proof.** By Equation 2, the theorem holds for all transverse perturbations $C' \cap D'$ of $C$ and $D$. By Theorem 3, the numbers are the same for the stable intersection. 

5
4 Complements

Areas

The definition of intersection multiplicity in general involves areas. Whereas for transverse intersections the area is of a parallelogram, which is easy, finding the area of a general convex polygon can be challenging. For this, we present two tools: one lattice-based, and the other vector-based.

**Proposition** (Pick’s theorem). Let \( P \) be a polygon in \( \mathbb{R}^2 \) whose vertices are in \( \mathbb{Z}^2 \); denote by \( I(P) \) and \( B(P) \) the number of points of \( \mathbb{Z}^2 \) which are respectively interior to \( P \) and on its boundary (including vertices). Then \( A(P) = I(P) - \frac{1}{2} B(P) + 1 \).

This theorem, which is surprisingly hard to prove (and we won’t), can at least be verified in all the common cases. It also covers nicely some difficult uncommon ones, however. In order to use it for Bezout’s theorem, you need to draw the Newton polygon; if you’d rather work with the balanced graph alone, though, you can always do:

**Proposition 5.** Let \( V \) be a vertex of a tropical curve \( C \), with Newton polygon \( \mathcal{N}_V(C) \), outward primitive vectors \( \overrightarrow{u}_i \), and multiplicities \( m_i \). Then

\[
A(\mathcal{N}_V) = \frac{1}{2} \sum_{j > i} m_i m_j \| \overrightarrow{u}_i \times \overrightarrow{u}_j \|.
\]

This is clearly the analogue of Equation 3.

Self-intersection

One very interesting consequence of Theorem 3 is that a single curve \( C \) has a stable intersection with itself: the points are its vertices, and their multiplicities, as you can check using an appeal to Proposition 2 to get the relationships of the areas, are twice the areas of the corresponding divisions. This is apparently a feature of the as-yet undeveloped “tropical intersection theory”, where one considers tropical varieties as elements of a group in which the operation is intersection; the self-intersection version of Bezout’s theorem says simply that if \( C \) is tropical of degree \( d \), then \( C^2 = d^2 \) (interpreted appropriately).