1 Summary of research interests

My mathematical work has two directions. The first is the focus of my thesis: *geometric representation theory*, the pursuit of connections between representations of algebraic groups and the structure of certain algebraic varieties (or ind-varieties). The second is *category-theoretic algebra*, which is not the study of categories of algebraic objects (e.g. abelian categories) but rather the study of the algebra of category theory, such as coherence theorems ("all diagrams commute").

2 PhD thesis: twisted geometric Satake equivalence

We begin with a brief overview of the history of the Satake equivalence up to my thesis work [Rei11].

Previous work

Drinfeld proposed, based on evidence of Lusztig, that for any reductive complex algebraic group G the category $\mathbf{Rep}({}^L G)$ of representations of its *Langlands dual group* can be realized as the category $\mathbf{Sph}(\mathrm{Gr}_G)$ of certain *spherical perverse sheaves* on a (ind-)variety Gr_G called the *affine grassmannian* of G. This realization includes a symmetric tensor category structure on $\mathbf{Sph}(\mathrm{Gr}_G)$ mirroring the tensor product of representations, and was subsequently proven by Ginzburg [Gin95] and Mirković–Vilonen [MV07].

The proof given by Mirković–Vilonen uses a more general affine grassmannian called the *factorizable*, or *Beilinson–Drinfeld grassmannian* Gr_{G,X^n} over the cartesian powers of a smooth curve X. The factorizability property refers to the property that, in its simplest instance, is

$$\mathrm{Gr}_{G,X^2}|_{X^2 \setminus \Delta} \cong (\mathrm{Gr}_{G,X})^2|_{X^2\setminus\Delta} \qquad\qquad \mathrm{Gr}_{G,X^2}|_{\Delta} \cong \mathrm{Gr}_{G,X} \text{ (with } X \cong \Delta).$$

This property enabled the tensor category structure on $\mathbf{Sph}(\mathrm{Gr}_G)$ to be defined on a pair of perverse sheaves $\mathcal{F}$, $\mathcal{G}$ by spreading them out along $\mathrm{Gr}_{G,X}$ and then performing the *fusion* operation

$$\mathcal{F} * \mathcal{G} = j_{!*}(\mathcal{F} \boxtimes \mathcal{G})|_{\Delta}[-1]$$

(here, $j\colon X^2 \setminus \Delta \to X^2$). Gaitsgory later [Gai07] restated the Satake equivalence entirely in terms of this factorizability property:

Theorem. *The category $\mathbf{Sph}(\mathrm{Gr}_n)$ is equivalent to that, denoted $\mathbf{FRep}({}^L G)_{X^n}$ in [Rei11], of perverse sheaves $\mathcal{F}$ on X^n together with a* factorizable action *of ${}^L G$. This concept is exemplified for $n = 2$: $\mathcal{F}$ should have an action of ${}^L G$, $\mathcal{F}|_{X^2\setminus\Delta}$ an action of $({}^L G)^2$, and the restriction of the former to Δ_X must be the restriction to Δ_G in the latter. This equivalence sends convolution to tensor product.*

Finally, the setup of the Satake equivalence was generalized by Finkelberg–Lysenko [FL09] to include *twisting* by a root of unity q; they obtained an equivalence of spherical twisted sheaves with $\mathbf{Rep}(\check{G}_q)$ for some new dual group $\check{G}_q$. This twisting is defined using sheaves on a certain line bundle (the *determinant bundle*) on Gr_G having prescribed monodromy around the fibers.

Thesis results

The main results of my thesis have two parts: the first is the construction of a framework, that of *twisting by a gerbe*, for the twisted Satake equivalence of Finkelberg–Lysenko, and the identification of exactly those gerbes that are suitable. (We leave the term "gerbe" unspecified in this summary.) The main result in this direction can be stated as:

Theorem. *Let $\mathscr{G}$ be a gerbe on Gr_G, $\mathbf{Sph}(\mathscr{G})$ the category of spherical $\mathscr{G}$-twisted perverse sheaves. Then there is a twisted Satake-type equivalence (including the construction of a dual group $\check{G}_{\mathscr{G}}$) if and only if $\mathscr{G}$ comes from a* factorizable gerbe *$\{\mathscr{G}_n\}$ on Gr_{G,X^n}. Such gerbes form a 2-category whose objects are naturally identified with quadratic forms on the coweight lattice of G, invariant for the Weyl group of G, and valued in the multiplicative group $\mathbb{C}^\times$ of complex numbers. (In fact, any abelian group works.)*

The appearance of W-invariant quadratic forms is reminiscent of the concept of *level* in representations of Kac–Moody algebras and that also appears in the work of Beilinson–Drinfeld [BD] in the form of "critical twisting", i.e. at level $(1/2)\kappa$, where κ is the Killing form of G.

The second main result joins Gaitsgory's and Finkelberg–Lysenko's in a single statement. Given in imprecise terms, it is:

Theorem. *Let $\{\mathscr{G}_n\}$ be a factorizable gerbe on Gr_{G,X^n}, corresponding to a quadratic form Q. Then there are: a dual reductive group $\check{G}_Q$; a gerbe $\mathscr{Z}_n$ on X^n; a convolution product on the spherical $\mathscr{G}_n$-twisted perverse sheaves $\mathbf{Sph}(\mathscr{G}_n)$; and a tensor product on the $\mathscr{Z}_n$-twisted perverse sheaves with a factorizable action of $\check{G}_Q$, $\mathbf{FRep}(\mathscr{Z}_n)$. Finally, there is an equivalence*

$$\mathbf{Sph}(\mathscr{G}_n) \xrightarrow{\sim} \mathbf{FRep}(\mathscr{Z}_n)$$

sending convolution to tensor product.

Not only is the statement more general, but the proof uses new techniques for many of the intermediate results that are not themselves new work: the proof that $\mathbf{Sph}(\mathrm{Gr}_G)$ is a semisimple abelian category; that convolution preserves perverse sheaves; and that the above equivalence (defined in a standard way via a *fiber functor*) is truly invertible.

3 Concrete Satake equivalence

We suggest a research project along the following lines: a concrete realization of the above construction of ${}^LG$, based on ideas in the above mentioned proof of semisimplicity of $\mathbf{Sph}(\mathrm{Gr}_G)$. First, for simplicity let G be a simple algebraic group, with highest coroot λ with respect to some choice B of Borel subgroup having torus T. Thus, the adjoint representation of ${}^LG$ in its Lie algebra ${}^L\mathfrak{g}$ has highest weight λ.

Background

Some notation for the geometric situation is necessary. Under the (Mirković–Vilonen) Satake equivalence, highest-weight representations of ${}^LG$ correspond to certain simple spherical sheaves $\mathcal{J}(\mu)$ on Gr_G, indexed by dominant coweights of G. They are supported on the closures of orbits in Gr_G of the arc group $G(\mathbb{C}[[t]])$, which acts on the grassmannian (we will not discuss this here); these orbits are denoted Gr_G^μ and contain distinguished points t^μ corresponding to the grassmannian Gr_T. The grassmannian Gr_B also has special significance; its components are naturally subsets of Gr_G denoted S^μ for *arbitrary* coweights μ. Finally, the convolution product in $\mathbf{Sph}(\mathrm{Gr}_G)$ is defined (before constructing the fusion product) using a *convolution diagram*

$$\mathrm{Gr}_G \mathbin{\widetilde{\times}} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G\,.$$

Now, first recall the following representation-theoretic fact:

Lemma. The space of maps of ${}^LG$-representations ${}^L\mathfrak{g} \otimes {}^L\mathfrak{g} \to {}^L\mathfrak{g}$ is spanned by the Lie bracket. □

This suggests that one can naturally identify the Lie bracket on $\mathcal{J}(\lambda)$. More generally, one hopes to exhibit any representation of ${}^L\mathfrak{g}$ via an action of $\mathcal{J}(\lambda)$ on any $\mathcal{J}(\mu)$; that is, a map $\mathcal{J}(\mu) \otimes \mathcal{J}(\lambda) \to \mathcal{J}(\mu)$. It is known [Rei11, Lemma V.6.8] that the maps $\mathcal{J}(\mu) \otimes \mathcal{J}(\lambda) \to \mathcal{J}(\lambda)$ are spanned, as a vector space, by the components of dimension $(1/2) \dim \mathrm{Gr}_G^\lambda$ in the space

$$m^{-1}(t^\mu) \cap (\mathrm{Gr}_G^\mu * \mathrm{Gr}_G^\lambda) \subset \mathrm{Conv}_G .$$

Here, $\mathrm{Gr}_G^\mu * \mathrm{Gr}_G^\lambda \subset \mathrm{Conv}_G$ is a "convolution product" of $G(\widehat{\mathcal{O}})$-invariant subspaces of Gr_G. It is shown in [Rei11, Lemma V.1.8] that for sufficiently "large" μ, we have

$$m^{-1}(t^\mu) \cap (\mathrm{Gr}_G^\mu * \mathrm{Gr}_G^\lambda) \supset m^{-1}(t^\mu) \cap (S^\mu * \mathrm{Gr}_G^\lambda) \cong m^{-1}(t^0) \cap (S^0 * \mathrm{Gr}_G^\lambda) \cong S^0 \cap \mathrm{Gr}_G^\lambda, \qquad (1)$$

where the last set is known to have pure dimension $(1/2) \dim \mathrm{Gr}_G^\lambda$.

Conjecture

It follows from the geometric Satake equivalence that the components C_α of $S^0 \cap \mathrm{Gr}_G^\lambda$ correspond naturally to a frame in the Cartan subalgebra of ${}^L\mathfrak{g}$ consisting of its simple coroot spaces. Suppose that (1) holds for μ; a map $\mathcal{J}(\mu) \otimes \mathcal{J}(\lambda) \to \mathcal{J}(\lambda)$ given by a combination $\sum a_\alpha C_\alpha$ should be a representation in which the coroot α acts on the μ-weight vector by the number $a_\alpha \langle \mu, \alpha \rangle$. This suggests the following conjecture:

Conjecture 1. *If (1) holds for μ, the map $\mathcal{J}(\mu) \otimes \mathcal{J}(\lambda) \to \mathcal{J}(\mu)$ which is given by the combination $\sum C_\alpha$ with all coefficients equal to 1 corresponds under the Satake equivalence to the structure map of the resulting representation.*

According to the Lemma, this conjecture should not hold for $\mu = \lambda$ unless ${}^L\mathfrak{g}$ has rank 1. Nonetheless, we also seek a geometric proof of the Lemma, and thus a geometric construction of the Lie bracket on ${}^L\mathfrak{g}$. Finally, we hope to discover how the conjecture may be generalized to cover all μ, even those insufficiently large.

4 Representations of the symmetric group

(Joint project with Igor Pak, Greta Panova, Brendan Rhoades)

In their paper [dCK81], Kazhdan and de Concini describe a topological approach to constructing bases for representations of the symmetric group S_n (and also for GL_n) which were previously described by Springer. These bases appear to have similar properties to the Kazhdan–Lusztig bases and are obtained in a geometric fashion as certain subvarieties of the flag variety.

The construction is as follows: for any unipotent matrix u of a given Jordan type (that is, any partition of n) we consider the set Fl_u of u-stable flags in the flag variety Fl_n. Each subminimal parabolic subgroup P_i of GL_n determines a variety Fl_{P_i} of partial flags over which Fl_n is a $\mathbb{P}^1$-bundle; there is an algebraically defined "antipode" map j_i on Fl_n acting as inversion on each fiber. Up to homotopy, j_i preserves Fl_u, determining an involution s_i of its top homology group, which is spanned by the classes of the irreducible components of Fl_u. These s_i's extend to a representation of S_n in $H_{\mathrm{top}}(\mathrm{Fl}_u)$, and these representations are exactly the irreducible ones.

The aim of our joint project is to give a fully algorithmizable realization of this construction, in particular to eliminate the homotopy ambiguity and precisely describe the representing matrices of the s_i's. The setup appears more amenable to computation than the Kazhdan–Lusztig basis, which we believe is different, though if not, the improved computability would be a significant result as well.

5 Coherence theorem

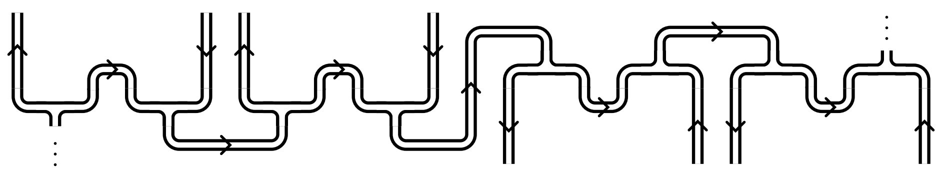
A practical problem familiar to every mathematician working with sheaves is that of checking commutativity of diagrams of "natural" morphisms of sheaves. Such morphisms are constructed, for example, from natural maps

$$f^* f_* \mathcal{F} \to \mathcal{F} \qquad\qquad \mathcal{G} \to f_* f^* \mathcal{G} \tag{2}$$

where $f\colon X \to Y$ is a map of spaces (for example, schemes) and $\mathcal{F}$, $\mathcal{G}$ are sheaves on X and Y respectively. These constructions also make sense in the derived category of sheaves, where the base-change theorems of cohomology can be expressed as saying that certain such natural morphisms are invertible, and then the class of useful natural morphisms is expanded to include their inverses. From these, arbitrarily complex diagrams are constructed and it is necessary to check that they commute, which is tedious because of the expectation that "nothing can go wrong".

We are currently completing a proof that, in fact, nothing can go wrong. Specifically, we consider a class of natural transformations among functors of sheaves that are constructed as chains of f^*, f_*, and their Verdier duals $f^!$ and $f_!$, which we call *geometric functors*. The natural transformations in question are all those that can be obtained by chains isomorphisms $(fg)_* \cong f_* g_*$ and so on; of those in (2) and their duals; and of the natural maps $f_! \to f_*$ from compactly supported cohomology to total cohomology; we call these *geometric natural transformations*.

Our method for dealing with the complexities of these transformations is by using *string diagram* notation for natural transformations. A general *connected* geometric natural transformation in this notation looks like:



where the 2-dimensional regions represent categories, vertical paths represent functors (going left to right), and horizontal turning points or intersections as natural transformations (going bottom to top) between the incident functors. In this diagram, the arrows also indicate whether the functor is an f_* (upward) or f^* (downward). We have shown:

Lemma. Every geometric natural transformation (involving only $*$ functors, or by duality only ! functors) is equal to one in the form of the above diagram.

We have also been able to show:

Theorem. Let F and G be any two geometric functors, necessarily containing only $*$ functors, and let $\phi, \psi\colon F \to G$ be any geometric natural transformations obtained only as compositions of the maps in (2), or of inverses of such compositions. If all the maps in G are quotient maps, and all the maps in F are immersions, then $\phi = \psi$.

We hope to determine the obstruction to extending this theorem to the derived category and to include ! functors as well.

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