The dual group for a twisted Satake equivalence and quadratic forms from gerbes

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- There is another seemingly formal construction of a dual group, due again to Lusztig, that generalizes ${}^L G$.
- We will describe how it can also be anticipated on geometric grounds.

Outline

1. The Langlands/Lusztig dual group
   - Root data
   - Cartan data and quadratic forms
   - Quadratic forms and the dual group
   - A computation

2. Quadratic forms and factorizable gerbes
   - The grassmannian and factorizability
   - Factorizable gerbes

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- The root data refines the root system so as to distinguish groups related in such a way.

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which indeed pairs with any weight via the inner product $(\bullet, \check{\alpha})$. The key property of root data is that Ψ^* and Ψ_* are closed under the *simple reflections* generating the *Weyl group* W:

$$s_\alpha \colon X_* \to X_*, \quad \lambda \mapsto \lambda - \langle \alpha, \lambda \rangle \lambda.$$

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- Since there's a 2-1 map $\mathrm{SL}_2 \to \mathrm{PGL}_2$, it's convenient to identify $X_{*,\mathrm{SL}_2} = 2\mathbb{Z} \subset \mathbb{Z} = X_{*,\mathrm{PGL}_2}$.

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- The pairing of a root and a coroot is supposed to come from an integral bilinear form $(i, j) \mapsto i \cdot j \in \mathbb{Z}$ on $\mathbb{Z}I$:

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- The bilinear form itself is supposed to come from an integral *quadratic* form $i \mapsto f(i) \in \mathbb{Z}$ (with $f(i) > 0$),

$$i \cdot j = f(i + j) - f(i) - f(j) \qquad f(i) = \tfrac{1}{2} i \cdot i.$$

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To get this, we expand out the invariance equation

$$f(i) = f(s_j i) = f(i - \langle \psi^*(i), \psi_*(j) \rangle j)$$

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Using the quadratic property:

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Now replace λ by $\lambda + j$; we still have $\langle \psi^*(i), \psi_*(\lambda + j) \rangle = 2$, so the same equation applies. The left-hand side is linear in λ, so $i \cdot j = 0$.

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- The new "dual" root data $(Y^*, \phi^*, Y_*, \phi_*)$ are:

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- Then we have $l_i = \operatorname{ord} Q(\psi_*(i))$ and

$$Y^* = \{\lambda \in X_* \mid \forall i \in I, B(\lambda, \psi_*(i)) = 1\} = \ker B.$$

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- So the simple reflections in Φ^* are the ones inherited from Ψ_*. Since Q is W-invariant, it follows that Φ^* is closed under simple reflections (also Φ_*).

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- That proves 1 when $\langle \check{\alpha}, \lambda \rangle = 1$. Finally, both sides are linear in λ, so it follows by induction for all $\langle \check{\alpha}, \lambda \rangle$.

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- When B (hence β) is trivial, we expect to recover the usual Langlands dual, so the natural dual for a general B is $\operatorname{coker} \beta$.

Why this dual group? (Torus case)

We will derive the formulas for the dual root data from some reasonable hypotheses, some of which come from geometry.

- First consider what dual group we should associate with a torus and a $\mathbb{C}^*$-valued quadratic form on its coweights X_*.
- The associated bilinear form B is a map $X_* \to X^* \otimes \mathbb{C}^*$ and thus induces

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- The weights of $\operatorname{coker} \beta$ are what we have called $\ker B$.

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- Suppose q has order r; then:

$$\ker B = \{n \mid \forall m, r|2nm\} = \begin{cases} 0 & r = \infty \\ r\Lambda & r \text{ odd} \\ \frac{1}{2}r\Lambda & r \text{ even} \end{cases}$$

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- and that they are equal when $\operatorname{ord}_2 r = 2$. The inclusions all have index 2.
- Since these lattices should be the weights of the dual groups $\check{\mathrm{SL}}_2$ and $\check{\mathrm{PGL}}_2$, they are respectively PGL_2 and SL_2 when $\operatorname{ord}_2 r \leq 1$, and are reversed when $\operatorname{ord}_2 r \geq 3$.

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- Going back to the numbers in the lattice computations, we also find that the root of the dual group is in all cases equal to r times the coroot of PGL_2.
- Note that $r = \operatorname{ord} Q(\check{\alpha})$, if $\check{\alpha}$ is that coroot. The same holds (with more case analysis) for SL_2.

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Theorem

If two reductive groups have the same weights (after choosing a maximal torus), the same dominant weights (after choosing Borel subgroups), and all the Levi subgroups of one map to the other, then these maps extend to an isomorphism.

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- We assumed that the duals of these groups had rank 1 (rather than 0); again, this follows from geometric facts.

Outline

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- It is a sequence of spaces $\mathrm{Gr}_{T,n}$ pieced together from copies of $\mathbb{A}^n$. Each one has infinitely many irreducible components $\mathrm{Gr}_{T,n}^{\lambda_1,\dots,\lambda_n}$ corresponding to n coweights of T.

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- For $n = 3$, components $\mathrm{Gr}_{T,3}^{\lambda_1,\lambda_2,\lambda_3}$ and $\mathrm{Gr}_{T,3}^{\mu_1,\mu_2,\mu_3}$ intersect along $\{x_i = x_j\}$ (the third coordinate being x_k) if:

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- Linearity suggests that we consider $B(\lambda, \mu)$ associated with the diagonal Δ in $\mathrm{Gr}_{T,2}^{\lambda,\mu} \cong \mathbb{A}^2$.
- A complex number attached to Δ suggests the *monodromy* of a locally constant sheaf on some punctured neighborhood of Δ.

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- The fact that “some” punctured neighborhood is nonspecific suggests that this sheaf is part of a gluing setup.
- (We glue something on the diagonal to something off the diagonal using a small neighborhood whose exact nature is unimportant.)
- It turns out that when the gluing data is a sheaf (rather than a section of a sheaf) the glued object is a higher-categorical concept called a *gerbe*.

Factorizable gerbes

- We will take gerbes as a black box subject to the following useful lemma: given gerbes $\mathscr{G}$ and $\mathscr{H}$ on some space X, and a smooth, codimension-1 subvariety Z:

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- (Intuition: a gerbe is like a principal bundle for the "group" of one-dimensional locally constant sheaves. Gluing copies of this group across Z requires giving an "element" on a punctured neighborhood; locally constant sheaves on $\mathbb{C} - 0$ are classified by their monodromy.)

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- Since $\mathrm{Gr}_{T,2}^{\lambda,\mu} \cong \mathrm{Gr}_{T,1}^{\lambda} \times \mathrm{Gr}_{T,1}^{\mu}$ (stronger than factorizability), the product on the right side of the above equation is defined *across* the diagonal too. So the isomorphism there has a monodromy.

Bilinear forms from factorizable gerbes

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- Note that the left-hand side, restricted to $x_1 = x_2$, gives $\mathscr{G}_2^{\lambda+\mu,\nu}$. The right-hand side has a complication we will gloss over.

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- Let's find the order of ϕ around the diagonals $x_1 = x_3$ and $x_2 = x_3$. We use the slightly more informative factorization
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- The monodromies multiply around the intersection:
$$B(\lambda + \mu, \nu) = B(\lambda, \nu) B(\mu, \nu).$$